

Fault-Tolerant Spanners for Doubling Metrics: Better and Simpler

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Abstract

In STOC'95 Arya et al. [2] conjectured that for any constant dimensional n -point Euclidean space, a $(1 + \epsilon)$ -spanner with constant degree, diameter $O(\log n)$ and weight $O(\log n) \cdot \omega(MST)$ can be built in $O(n \log n)$ time. Recently Elkin and Solomon [8] (technical report, April 2012) proved this conjecture of Arya et al. in the affirmative. In fact, the proof of [8] is more general in two ways. First, it applies to arbitrary doubling metrics. Second, it provides a complete tradeoff between the three involved parameters that is tight (up to constant factors) in the entire range.

Subsequently, Chan et al. [5] (technical report, July 2012) provided another proof for Arya et al.'s conjecture, which is simpler than the proof of Elkin and Solomon [8]. Moreover, Chan et al. [5] also showed that one can build a fault-tolerant (FT) spanner with similar properties. Specifically, they showed that there exists a k -FT $(1 + \epsilon)$ -spanner with degree $O(k^2)$, diameter $O(\log n)$ and weight $O(k^3 \cdot \log n) \cdot \omega(MST)$. The running time of the construction of [5] was not analyzed.

In this note we improve the results of Chan et al. [5], using a simpler proof. Specifically, we present a simple proof which shows that a k -FT $(1 + \epsilon)$ -spanner with degree $O(k^2)$, diameter $O(\log n)$ and weight $O(k^2 \cdot \log n) \cdot \omega(MST)$ can be built in $O(n \cdot (\log n + k^2))$ time. Similarly to the constructions of [8] and [5], our construction applies to arbitrary doubling metrics. However, in contrast to the construction of Elkin and Solomon [8], our construction fails to provide a complete (and tight) tradeoff between the three involved parameters. The construction of Chan et al. [5] has this drawback too.

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1 Introduction

Consider a set P of n points in \mathbb{R}^d and a number $t \geq 1$, and let $G = (P, E)$ be a graph in which the weight $\omega(x, y)$ of each edge $e = (x, y) \in E$ is equal to the Euclidean distance $\|x - y\|$ between x and y . The graph G is called a t -spanner for P if for every $p, q \in P$, there is a path in G between p and q whose weight (i.e., the sum of all edge weights in it) is at most $t \cdot \|p - q\|$. Such a path is called a t -spanner path. The problem of constructing Euclidean spanners has been studied intensively (see, e.g., [6, 11, 1, 2, 12, 7]).

Euclidean spanners find applications in geometric approximation algorithms, network topology design, distributed systems, and other areas. In many applications it is required to construct a $(1 + \epsilon)$ -spanner $G = (P, E)$ that satisfies some useful properties. First, the spanner should contain $O(n)$ (or nearly $O(n)$) edges. Second, its *weight*¹ $\omega(G) = \sum_{e \in E} \omega(e)$ should not be much greater than the weight $\omega(MST(P))$ of the minimum spanning tree $MST(P)$ of P . Third, its (*hop*-) *diameter* $\Lambda(G)$ should be small, i.e., for every $p, q \in P$ there should be a path Π in G that contains at most $\Lambda(G)$ edges and has weight $\omega(\Pi) = \sum_{e \in E(\Pi)} \omega(e) \leq (1 + \epsilon) \cdot \|p - q\|$. Fourth, its (*maximum*) *degree* $\deg(G)$ should be small.

In STOC'95 [2], Arya et al. showed that for any set of n points in \mathbb{R}^d one can build in $O(n \log n)$ time a $(1 + \epsilon)$ -spanner with constant degree, diameter $O(\log n)$ and lightness $O(\log^2 n)$. They conjectured that one can obtain in the same time a spanner with constant degree, and logarithmic diameter and lightness.

This conjecture of Arya et al. [2] was resolved in the affirmative recently by Elkin and Solomon [8]. In fact, the result of [8] is more general in two ways. First, it applies to arbitrary *doubling metrics*.² Second, it provides a complete tradeoff between the involved parameters. Specifically, Elkin and Solomon [8] showed that one can build in $O(n \log n)$ time a $(1 + \epsilon)$ -spanner with degree $O(\rho)$, diameter $O(\log_\rho n + \alpha(\rho))$ and lightness $O(\rho \cdot \log_\rho n)$, where $\rho \geq 2$ is an integer parameter and α is the inverse Ackermann function. Due to lower bounds of Dinitz et al. [7], this tradeoff is tight (up to constant factors) in the entire range.

Later, Chan et al. [5] provided a simpler proof for Arya et al.'s conjecture. Moreover, they strengthened their construction to be *fault-tolerant* (FT).³ Specifically, they showed that there exists a k -FT $(1 + \epsilon)$ -spanner with degree $O(k^2)$, diameter $O(\log n)$ and lightness $O(k^3 \cdot \log n)$. The running time of the construction of [5] was not analyzed; we remark that a naive implementation requires quadratic time.

In this note we improve the results of Chan et al. [5], using a simpler proof. Specifically, we present a simple proof which shows that a k -FT $(1 + \epsilon)$ -spanner with degree $O(k^2)$, diameter $O(\log n)$ and lightness $O(k^2 \cdot \log n)$ can be built in $O(n \cdot (\log n + k^2))$ time, for any integer $1 \leq k \leq n - 2$. Similarly to the constructions of [8] and [5], our construction applies to arbitrary doubling metrics. However, in contrast to the construction of Elkin and Solomon [8], our construction fails to provide a complete (and tight) tradeoff between the involved parameters. The construction of Chan et al. [5] has this drawback too.

1.1 Our and Previous Techniques. The starting point of the construction of Elkin and Solomon [8] is to “shortcut” an MST-tour (or any Hamiltonian path with constant lightness) of the metric, using a 1-spanner construction from [13] that has constant degree and logarithmic diameter and lightness. Chan et al. [5] suggested a similar idea, which involves shortcutting the net-tree of [9, 3] (instead of the MST-tour) using a generalized 1-spanner construction for tree metrics from the same paper [13]. Each of the two approaches has advantages and disadvantages over the other one. The MST-tour has constant lightness, but it may blow up distances between points. On the other hand, distances between points in the net-tree can be controlled, but the lightness of the net-tree may be logarithmic (as in 1-dimensional Euclidean metrics where the points are uniformly spaced). In both approaches the main difficulty is to bound the degree of the spanner. The argument of [8] is more involved, mainly because one has to overcome the obstacle that distances between points in the MST-tour may blow up. On the bright side, since the MST-tour has constant lightness, it is possible to get a complete (and tight) tradeoff between all

¹For convenience, we will henceforth refer to the normalized notion of weight $\Psi(G) = \frac{\omega(G)}{\omega(MST(P))}$, which we call *lightness*.

²The *doubling dimension* of a metric is the smallest value d such that every ball B in the metric can be covered by at most 2^d balls of half the radius of B . A metric is called *doubling* if its doubling dimension is constant.

³A spanner G for a point set P (or for any metric) is called a k -FT t -spanner (or simply k -FT spanner, if t is clear from the context), for any $t \geq 1$ and $1 \leq k \leq n - 2$, if for any subset $F \subseteq P$ with $|F| \leq k$, $G \setminus F$ is a t -spanner for $P \setminus F$.

parameters in the entire range. The argument of [5] is simpler due to the “nice” structure of the net-tree. On the negative side, since the lightness of the net-tree may be logarithmic, extending the basic tradeoff (constant degree, logarithmic diameter and lightness) to the entire range is doomed to failure.

We follow the ideas of [8] and [5], but propose a simpler argument. Instead of shortcutting the MST-tour or the net-tree, we shortcut the underlying tree of the bounded degree spanner of Gottlieb and Roditty [10]. More specifically, similarly to the argument of [5], we shortcut the *light subtrees* of the tree (those with distance scales less than Δ/n , where Δ is the aspect ratio⁴ of the metric). Since the spanner of [10] has bounded degree, this *immediately* gives rise to the desired bounds. Thus our argument bypasses the main difficulties that arose in [8] and [5] on the way to reducing the degree. Another advantage of our argument is that it extends to provide a FT spanner in a *straightforward* way. On the other hand, the argument of Chan et al. [5] for extending their basic spanner construction to provide a FT one is quite elaborate and non-trivial, and is based on the ICALP’12 paper [4] of these authors.

2 The Basic Spanner Construction

In this section we provide a simple proof for the conjecture of Arya et al. [2].

Let $M = (P, \delta)$ be an n -point doubling metric. A $(1+\epsilon)$ -spanner H for M is called a *tree-like spanner*, if it “contains” a tree T (referred to as a *tree-skeleton* of H) that satisfies the following conditions:

1. Each vertex v of T is assigned a representative point $r(v) \in P$.
There is a 1-1 correspondence between the points of P and the representatives of the leaves of T . Each internal vertex is assigned a unique representative. (Thus, each point of P will be the representative of at most two vertices of T . In particular, there are at most $2n$ vertices in T .)
2. Each vertex v of T has a *radius* $rad(v)$. The radii of vertices decrease geometrically with the level; the radius of the root of T is the maximum inter-point distance, and the radii of leaves is at least the minimum inter-point distance (assumed to be 1). In particular, the depth of T is $O(\log \Delta)$, where Δ is the aspect ratio of M . For any vertex v in T , the metric distance between its representative $r(v)$ and the representative of any vertex that belongs to the subtree T_v of T rooted at v is at most $O(rad(v))$; hence the metric distance between every pair of representatives from T_v is $O(rad(v))$. In particular, the weight of all edges between a vertex v and its children in T is $O(rad(v))$; hence the tree distance between every pair of vertices from T_v is $O(rad(v))$. (The weight of an edge (x, y) is given by the metric distance $\delta(r(x), r(y))$ between the respective representatives.)
3. For any two points $p, q \in P$, there is a $(1+\epsilon)$ -spanner path in H between p and q that is composed of three consecutive parts: (a) a path ascending the edges of T , from the leaf u whose representative is p to some ancestor u' of u in T ; (b) a single *lateral* edge (u', v') ; (c) a path descending the edges of T , from v' to the leaf v whose representative is q . (Each edge (u, v) between a pair of vertices of T is translated into an edge $(r(u), r(v))$ in H of weight $\delta(r(u), r(v))$.) The weight of the ascending (respectively, descending) path is at most $O(rad(u'))$ (resp., $O(rad(v'))$). The weight $\delta(r(u'), r(v'))$ of the lateral edge is $\Omega(1/\epsilon) \cdot (rad(u') + rad(v'))$, hence it *dwarfs* the weights of the ascending and descending paths (i.e., it is larger than them by a factor of $\Omega(1/\epsilon)$).

Gottlieb and Roditty [10] proved the following theorem.

Theorem 2.1 ([10]) *For any n -point doubling metric $M = (P, \delta)$, and any constant $\epsilon > 0$, one can build in $O(n \log n)$ time a $(1+\epsilon)$ -spanner H and a tree-skeleton T for H (as defined above), with $\deg(H) = O(1)$.*

The tree T and the corresponding tree-like spanner H of Gottlieb and Roditty [10] are similar to the net-tree and the corresponding net-tree spanner [9, 3]. In particular, the standard analysis of [3] (see also [5]) implies that the lightness of the net-tree spanner is logarithmic. Using the same considerations it can be shown that the tree-like spanner H of [10] has logarithmic lightness as well.

⁴The *aspect ratio* of a metric is defined as the ratio between the maximum and minimum inter-point distance in it.

Similarly to the net-tree spanner of [9, 3], the tree-like spanner of Gottlieb and Roditty [10] may have a large diameter. More specifically, the diameter of the spanner is linear in the depth of the underlying tree. To reduce the diameter, we employ the following tree-shortcutting theorem from [13].

Theorem 2.2 (Theorem 3 in [13]) *Let T be an arbitrary n -vertex tree, and denote by M_T the tree metric induced by T . One can build in $O(n \log_\rho n)$ time, for any integer $\rho \geq 2$, a 1-spanner for M_T with $O(n)$ edges, degree at most $\deg(T) + 2\rho$ and diameter $O(\log_\rho n + \alpha(\rho))$.*

Next, we describe a simple spanner construction H^* , which coincides with Arya et al.'s conjecture.

We start by building the spanner H and its tree-skeleton T that are guaranteed by Theorem 2.1. Note that T contains at most $2n = O(n)$ vertices. Next, following an idea of [5], we shortcut all the *light subtrees* of T , i.e., the subtrees of T with distance scales less than Δ/n . More specifically, denote the light subtrees of T by T_1, \dots, T_ℓ . For every subtree T_i , we employ Theorem 2.2 in the particular case $\rho = O(1)$ to build a 1-spanner G_i for the tree metric M_{T_i} induced by T_i with bounded degree and logarithmic diameter. Notice that the edge weights of G_i are assigned according to the distance function of M_{T_i} . The 1-spanner G_i is then converted into a graph G_i^* over the point set P , by replacing each edge (u, v) of G_i (where u and v are vertices of T_i) with the edge $(r(u), r(v))$ between their respective representatives. Finally, let H^* be the spanner obtained from the union of the $\ell + 1$ graphs $H, G_1^*, \dots, G_\ell^*$.

Denote by n_i the number of vertices in the subtree T_i , for each $1 \leq i \leq \ell$. We have $\sum_{i=1}^\ell n_i \leq 2n$.

By Theorems 2.1 and 2.2, the time needed to build H^* is $O(n \log n) + \sum_{i=1}^\ell O(n_i \log n_i) = O(n \log n)$.

By Theorem 2.1, $\deg(T) \leq \deg(H) = O(1)$. Also, theorem 2.2 yields $\deg(G_i) \leq \deg(T_i) + 2\rho \leq \deg(T) + O(1) = O(1)$, for each $1 \leq i \leq \ell$. Since each point of P is assigned as the representative of at most two vertices of T , we have $\deg(H^*) \leq \deg(H) + 2 \cdot \max\{\deg(G_i) \mid 1 \leq i \leq \ell\} = O(1)$.

The lightness of H is $O(\log n)$. For each subtree T_i and any pair u, v of vertices in T_i , the metric distance between $r(u)$ and $r(v)$ is at most linear in the distance scale $O(\Delta/n)$ of T_i . Hence, the weight of each edge of G_i^* is $O(\Delta/n)$. The total weight of all graphs G_1^*, \dots, G_ℓ^* (there are overall $O(n)$ edges in these graphs) is therefore $O(n \cdot (\Delta/n)) = O(\Delta) = O(\omega(MST(M)))$. Thus, the lightness of H^* is $O(\log n)$.

Finally, we show that H^* is a $(1 + \epsilon)$ -spanner for M with diameter $O(\log n)$. Consider any pair $p, q \in P$ of points, and let u, v be their respective leaves in T . Since T is a tree-skeleton of H , there is a $(1 + \epsilon)$ -spanner path in H between p and q that is composed of three consecutive parts: (a) a path ascending the edges of T from u to u' ; (b) a lateral edge (u', v') ; (c) a path descending the edges of T from v' to v . Let T_i and T_j be the light subtrees to which u and v belong, respectively. (It is possible that $T_i = T_j$.) Let \tilde{u} (respectively, \tilde{v}) be the last vertex that belongs to T_i (resp., T_j) on the path in T from u to u' (resp., v to v'). (It is possible that $\tilde{u} = u'$ and/or $\tilde{v} = v'$.) We use the graph G_i^* (respectively, G_j^*) to shortcut the path from u to \tilde{u} (resp., v to \tilde{v}); the resulting path has the same (or smaller) weight, but only $O(\log n)$ edges. Also, the path from \tilde{u} to u' (respectively, \tilde{v} to v') has only $O(\log n)$ edges as well. It follows that there is a $(1 + \epsilon)$ -spanner path between p and q in H^* that has at most $O(\log n)$ edges.

Theorem 2.3 *For any n -point doubling metric $M = (P, \delta)$, and any constant $\epsilon > 0$, a $(1 + \epsilon)$ -spanner H^* with constant degree, and diameter and lightness $O(\log n)$ can be built in $O(n \log n)$ time.*

3 The Fault-Tolerant Spanner Construction

In this section we present a construction H_{FT}^* of FT spanners that is obtained by a straightforward modification of the basic spanner construction H^* from the previous section.

To get a k -FT spanner, we will assign $2k + 1$ representatives (instead of a single representative as in the basic construction) to any vertex v of the tree-skeleton T . We would like these representatives to be close (in terms of metric distance) to the original representative $r(v)$ of v in the basic construction. Since the radii of vertices in T decrease geometrically with the level, the original representative $r(x)$ of any descendant x of v in T is close to $r(v)$, and so it may serve as a representative of v in the k -FT spanner.

Denote by $D(v)$ the set of all descendants of a vertex v in T (including v itself). Let $D^*(v)$ be a subset of $2k + 1$ vertices from $D(v)$ with smallest level in T (i.e., those of shortest hop-distance from v

in T); if $|D(v)| \leq 2k + 1$, we have $D^*(v) = D(v)$. Let $R^*(v)$ be the set of original representatives of the vertices in $D^*(v)$, i.e., $R^*(v) = \{r(x) \mid x \in D^*(v)\}$; note that $|R^*(v)| \leq |D^*(v)| \leq 2k + 1$. Since each point of P is assigned as the representative of at most two vertices of T , we have $|R^*(v)| \geq \lceil |D^*(v)|/2 \rceil$. Hence, either $R^*(v)$ contains the representatives of all the vertices in the subtree T_v of T rooted at v or it must hold that $|D^*(v)| = 2k + 1$, and in the latter case we have $|R^*(v)| \geq k + 1$. Observe that each vertex in T belongs to sets $D^*(w)$ of at most $2k + 1$ vertices w in T , which implies that each point of P belongs to sets $R^*(w)$ of at most $4k + 2$ vertices w in T .

The FT spanner construction H_{FT}^* is obtained from the basic construction H^* in the obvious way. Specifically, for each edge $(r(u), r(v))$ in the basic spanner construction H^* (which is associated with the edge (u, v) between the respective vertices of T), our FT spanner construction H_{FT}^* will contain a complete bipartite graph between $R^*(u)$ and $R^*(v)$.

It is easy to see that, given the tree-skeleton T and the basic spanner construction H^* , the FT spanner construction H_{FT}^* can be built in $O(k^2 \cdot n)$ time. The overall running time is therefore $O(n \cdot (\log n + k^2))$.

Consider an arbitrary point $p \in P$, and let w be a vertex in T such that $p \in R^*(w)$. For every edge (w, x) (associated with an edge $(r(w), r(x))$ of H^*) that is incident on w in the basic construction, p is connected to all $O(k)$ points of $R^*(x)$ in H_{FT}^* . Since $\deg(H^*) = O(1)$, this contributes $O(k)$ units to the degree of p . Since p may belong to sets $R^*(w)$ of at most $O(k)$ vertices w in T , it follows that the degree of p in H_{FT}^* is at most $O(k^2)$. Hence $\deg(H_{FT}^*) = O(k^2)$.

Observe that each edge $(r(u), r(v))$ of H^* is replaced by $O(k^2)$ edges of roughly the same weight in H_{FT}^* . More specifically, the weight of each of these $O(k^2)$ edges is greater than the weight $\delta(r(u), r(v))$ of the original edge $(r(u), r(v))$ by an additive factor of $O(\text{rad}(u) + \text{rad}(v))$. Since the basic construction H^* has constant degree, it follows that $\omega(H_{FT}^*) \leq O(k^2) \cdot (\omega(H^*) + \sum_{v \in T} \text{rad}(v))$. Using the standard analysis of [3, 5] we conclude that the lightness of the FT spanner construction H_{FT}^* is $O(k^2 \cdot \log n)$.

Finally, we show that for any pair $p, q \in P \setminus F$ of functioning points (where $F \subseteq P$ is an arbitrary set of non-functioning points with $|F| \leq k$), there is a $(1 + O(\epsilon))$ -spanner path in $H_{FT}^* \setminus F$ with $O(\log n)$ edges. Let u and v be the respective leaves of p and q in T , and consider the $(1 + \epsilon)$ -spanner path Π between p and q in H^* . It has $O(\log n)$ edges, and it is composed of three consecutive parts: (a) a path $\Pi_u = (u = u_0, \dots, u_i = u')$ from $u = u_0$ to $u_i = u'$, such that $u_{i'}$ is an ancestor of $u_{i'-1}$ in T , for each $1 \leq i' \leq i$; (b) a lateral edge (u', v') ; (c) a path $\Pi_v = (v' = v_0, \dots, v_j = v)$ from $v' = v_0$ to $v_j = v$, such that $v_{j'}$ is a descendant of $v_{j'-1}$ in T , for each $1 \leq j' \leq j$. Note that $p \in R^*(u)$. More generally, for each vertex $u_{i'}$ of Π_u , either $p \in R^*(u_{i'})$ or $|R^*(u_{i'})| \geq k + 1$. Thus $R^*(u_{i'})$ contains at least one functioning point, for each $0 \leq i' \leq i$. Take $f(u_{i'})$ to be p whenever possible, i.e., for every $0 \leq i' \leq i$ such that $p \in R^*(u_{i'})$; otherwise take $f(u_{i'})$ to be an arbitrary functioning point from $R^*(u_{i'})$. (Note that $f(u) = p$.) Similarly, $R^*(v_{j'})$ contains at least one functioning point, for each $0 \leq j' \leq j$. Take $f(v_{j'})$ to be q whenever possible; otherwise take $f(v_{j'})$ to be an arbitrary functioning point from $R^*(v_{j'})$. (Note that $f(v) = q$.) By replacing each vertex x of Π with the respective functioning point $f(x)$, we obtain a path Π_{FT} in $H_{FT}^* \setminus F$ between $f(u) = p$ and $f(v) = q$ with the same number $O(\log n)$ of edges. Since the path Π_{FT} is a perturbation of the original $(1 + \epsilon)$ -spanner path Π (and the radii of vertices decrease geometrically with the level), it incurs only a small additional cost of $O(\text{rad}(u') + \text{rad}(v')) = O(\epsilon) \cdot \delta(r(u'), r(v')) = O(\epsilon) \cdot \delta(p, q)$. In other words, the stretch increases by an additive factor of $O(\epsilon)$, from $1 + \epsilon$ to $1 + O(\epsilon)$. One can reduce the stretch back to $1 + \epsilon$, at the expense of increasing the other parameters by some function of ϵ .

Theorem 3.1 *For any n -point doubling metric $M = (P, \delta)$, any constant $\epsilon > 0$, and any integer $1 \leq k \leq n - 2$, a k -FT $(1 + \epsilon)$ -spanner H_{FT}^* with degree $O(k^2)$, diameter $O(\log n)$ and lightness $O(k^2 \cdot \log n)$ can be built in $O(n \cdot (\log n + k^2))$ time.*

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